
UNIT 4 BASICS OF PROBABILITY

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4.0 INTRODUCTION

In our daily lives, we face many situations when we are unable to forecast the future with complete certainty. That is, in many decisions, uncertainty is faced. Need to cope up with uncertainty leads to the study and use of the probability theory. The first attempt to give quantitative measure of probability was made by Galileo (1564-1642), an Italian mathematician, when he was answering the following question on the request of his patron, the Grand Duke of Tuscany, who wanted to improve his performance at the gambling tables: "With three dice a total of 9 and 10 can each be produced by six different combinations, and yet experience shows that the number 10 is oftener thrown than the number 9?" To the mind of his patron the cases were (1, 2, 6), (1, 3, 5), (1, 4, 4), (2, 2, 5), (2, 3, 4), (3, 3, 3) for 9 and (1, 3, 6), (1, 4, 5), (2, 2, 6), (2, 3, 5), (2, 4, 4), (3, 3, 4) for 10 and hence he was thinking that why they do not occur equally frequently i.e. why their chances are not the same? Galileo makes a careful analysis of all the cases which can occur, and he showed that out of the 216 possible cases 27 are favourable to the appearance of the number 10 since permutations of (1, 3, 6) are (1, 3, 6), (1, 6, 3), (3, 1, 6), (3, 6, 1), (6, 1, 3), (6, 3, 1) i.e. number of permutations of (1, 3, 6) is 6; similarly, the number of permutations of (1, 4, 5), (2, 2, 6), (2, 3, 5), (2, 4, 4), (3, 3, 4) is 6, 3, 6, 3, 3 respectively and hence the total number of cases come out to be $6 + 6 + 3 + 6 + 3 + 3 = 27$ whereas the number of favourable cases for getting a total of 9 on three dice are $6 + 6 + 3 + 3 + 6 + 1 = 25$. Hence, this was the reason for 10 appearing oftener thrown than 9. But the first foundation was laid by the two mathematicians Pascal (1623-62) and Fermat (1601-65) due to a gambler's dispute in 1654, which led to the creation of a mathematical theory of probability by them. Later, important contributions were made by various researchers including Huyghens (1629 - 1695), Jacob Bernoulli (1654-1705), Laplace (1749-1827), Abraham De-Moivre (1667-1754), and Markov (1856-1922). Thomas Bayes (died in 1761, at the age of 59) gave an important technical result known as Bayes' theorem, published after his death in 1763, using which probabilities can be revised on the basis of some new information. Thereafter, the probability, an important branch of Statistics, is being used worldwide. We will start this unit with very elementary ideas. We will go step

by step clearing the basic ideas which are required to understand the probability. In this unit, we will first present the various terms which are used in the definition of probability and then we will give the classical definition of probability and simple problems. This will be followed by discussion on topics like Axioms of probability, conditional probability and Bayes' theorem.

4.1 OBJECTIVES

After studying this unit, you would be able to

- define and give examples of random experiment and trial;
- define and give examples of sample space, sample point and event;
- explain mutually exclusive cases
- explain the mathematical definition of probability;
- define and use axioms of probability;
- define and use the addition law, conditional probability and independent events;
- define and use Bayes' theorem;
- solve problems of probability based in addition law, conditional probability, independent events and Bayes' theorem.

4.2 SAMPLE SPACES AND EVENTS

In this section, first, we will define the terms Random Experiment and Trial, which form the basis of probability theory. This will be followed by a discussion on sample spaces and events.

Random Experiment: An experiment in which all the possible outcomes are known in advance, but we cannot predict as to which of them will occur when we perform the experiment, e.g. Experiment of tossing a coin is random experiment as the possible outcomes, "Heads" and "Tails" are known in advance, but which one will turn up is not known. Similarly, 'Throwing a die' and 'Drawing a card from a well shuffled pack of 52 playing cards' are the examples of random experiments.

Trial: Performing an experiment is called a trial, e.g.

- (i) Tossing a coin is a trial.
- (ii) Throwing a die is a trial.

Sample Space: Set of all possible outcomes of a random experiment is known as sample space and is usually denoted by S , and the total number of elements in the sample space is known as size of the sample space and is denoted by $n(S)$, for example:

- (i) If we toss a coin, then the sample space is $S = \{H, T\}$, where H and T denote Heads and Tails, respectively and $n(S) = 2$.
- (ii) If a die is thrown, then the sample space is $S = \{1, 2, 3, 4, 5, 6\}$ and $n(S) = 6$, as the die has six faces numbered 1, 2, 3, 4, 5, 6.
- (iii) If a coin and a die are thrown simultaneously, then the sample space is $S = \{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$ and $n(S) = 12$. where $H1$ denotes that the coin shows Heads and die shows 1, etc.

Note: Unless stated, the coin means an unbiased coin (i.e. the coin which favours neither Heads nor Tails)

- (iv) If a coin is tossed twice or two coins are tossed simultaneously, then the sample space is $S = \{HH, HT, TH, TT\}$, where HH means both the coins show heads, HT means the first coin shows heads and the second shows tails, etc. Here, $n(S) = 4$.
- (v) If a coin is tossed thrice or three coins are tossed simultaneously, then the sample space is $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$ and $n(S) = 8$.
- (vi) If a coin is tossed 4 times or four coins are tossed simultaneously, then the sample space is $S = \{HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, HTTH, THHT, THTH, TTHH, HTTT, THTT, TTHT, TTTH, TTTT\}$ and $n(S) = 16$.
- (vii) If a die is thrown twice or a pair of dice is thrown simultaneously, then the sample space is $S = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (5, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$ here, e.g., (1, 4) means the first die shows 1 and the second die shows 4. Here, $n(S) = 36$.
- (viii) If a family contains two children then the sample space is $S = \{B_1B_2, B_1G_2, G_1B_2, G_1G_2\}$ where B_i denotes that i^{th} birth is of boy, $i = 1, 2$, and G_i denotes that i^{th} birth is of girl, $i = 1, 2$. This sample space can also be written as $S = \{BB, BG, GB, GG\}$
- (ix) If a bag contains 2 red and 3 black balls, and
- One ball is drawn from the bag, then the sample space is $\{R_1, R_2, B_1, B_2, B_3\}$, where R_1, R_2 denote the two red balls and B_1, B_2, B_3 denote the three black balls in the bag.
 - Two balls are drawn one by one without replacement from the bag, then the sample space is

$$\{ R_1 R_2, R_1 B_1, R_1 B_2, R_1 B_3, \\ R_2 R_1, R_2 B_1, R_2 B_2, R_2 B_3, \\ B_1 R_1, B_1 R_2, B_1 B_2, B_1 B_3, \\ B_2 R_1, B_2 R_2, B_2 B_1, B_2 B_3, \\ B_3 R_1, B_3 R_2, B_3 B_1, B_3 B_2 \}$$

Note: It is very simple to write the above sample space – first write all other balls with R_1 and then with R_2 , then with B_1 and so on.

Remark 1: If a random experiment with x possible outcomes is performed n times, then the total number of elements in the sample is n^x i.e. $n(S) = n^x$, e.g. if a coin is tossed twice, then $n(S) = 2^2 = 4$; if a die is thrown thrice, then $n(S) = 3^6 = 216$.

Sample Point: Each outcome of an experiment is visualised as a sample point in the sample space. e.g.

- If a coin is tossed then getting head or tail is a sample point.
- If a die is thrown twice, then getting (1, 1) or (1, 2) or (1, 3) or ... or (6, 6) is a sample point.

Event: Set of one or more possible outcomes of an experiment constitutes what is known as event. Thus, an event can be defined as a subset of the sample space, e.g.

- i) In a die throwing experiment, event of getting a number less than 5 is the set $\{1, 2, 3, 4\}$, which refers to the combination of 4 outcomes and is a sub-set of the sample space $S = \{1, 2, 3, 4, 5, 6\}$.
- ii) If a card is drawn from a well-shuffled pack of playing cards, then the event of getting a card of a spade suit is $\{1s, 2s, 3s, \dots, 9s, 10s, Js, Qs, Ks\}$ where suffix s under each character in the set denotes that the card is of spade and J, Q and K represent jack, queen and king respectively.

Different Types of Events: The following are different types of events in Probability computation. However, we present here three simple but important types.

Simple Event: Simple event consists of only one sample point is a simple event. For example, getting the number 2 on roll of a die.

Composite Event: An event consisting of at least two sample points. For example, getting a number less than 5 is the set of getting numbers $\{1, 2, 3, 4\}$

Mutually Exclusive Events: Those events out of which only one event can occur at a time are called mutually exclusive events. For example, on throwing a dice, the events: Event 1: “Getting a number less than 3 with sample space $\{1, 2\}$ ”, Event 2: “Getting the number 4 or 5 with sample space $\{4, 5\}$ ” and Event 3: Getting a number 6; are mutually exclusive.

Let us take up some examples which define the events as subsets of sample space.

Example 1: If a fair die is thrown once, what is the event of:

- (i) getting an even number
- (ii) getting a prime number
- (iii) getting a number multiple of 3
- (iv) getting an odd prime
- (v) getting an even prime
- (vi) getting a number greater than 4
- (vii) getting a number multiple of 2 and 3

Solution: When a die is thrown, then the sample space is:

$$S = \{1, 2, 3, 4, 5, 6\}$$

- (i) Let E_1 be the event of getting an even number, $\therefore E_1 = \{2, 4, 6\}$
- (ii) Let E_2 be the event of getting a prime number, $\therefore E_2 = \{2, 3, 5\}$
- (iii) Let E_3 be the event of getting a number multiple of 3 $\therefore E_3 = \{3, 6\}$
- (iv) Let E_4 be the event of getting an odd prime, $\therefore E_4 = \{3, 5\}$
- (v) Let E_5 be the event of getting an even prime, $\therefore E_5 = \{2\}$
- (vi) Let E_6 be the event of getting a number greater than 4, $\therefore E_6 = \{5, 6\}$
- (vii) Let E_7 be the event of getting a number multiple of 2 and 3, $\therefore E_7 = \{6\}$

Example 2: If a pair of fair dice is thrown, what is the event of

- (i) getting a doublet
- (ii) getting sum as 11
- (iii) getting sum less than 5
- (iv) getting sum greater than 16
- (v) getting 3 on the first die
- (vi) getting a number multiple of 3 on second die

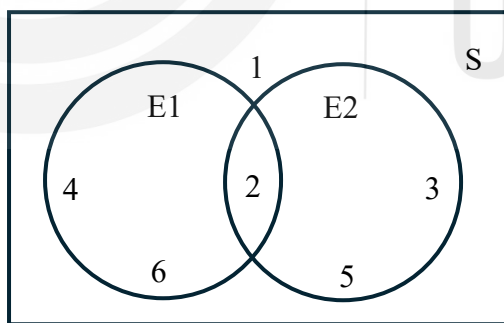
- (vii) getting a number multiple of 2 on first die and a multiple of 3 on second die.

Solution: The sample space, when two dice are thrown, is given in Sample Space point (vii) of this section.

- (i) Let E_1 be the event of getting a doublet. $\therefore E_1 = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$
- (ii) Let E_2 be the event of getting sum 11. $\therefore E_2 = \{(5, 6), (6, 5)\}$
- (iii) Let E_3 be the event of getting sum less than 5 i.e. sum can be 2 or 3 or 4 $\therefore E_3 = \{(1, 1), (1, 2), (2, 1), (1, 3), (3, 1), (2, 2)\}$
- (iv) Let E_4 be the event of getting sum greater than 16. $\therefore E_4 = \{\}$, i.e. E_4 is a null event.
- (v) Let E_5 be the event of getting 3 on the first die i.e. 3 on first die and second die may have any number $\therefore E_5 = \{(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)\}$
- (vi) Let E_6 be the event of getting a number multiple of 3 on second die i.e. first die may have any number and the second has 3 or 6. $\therefore E_6 = \{(1, 3), (2, 3), (3, 3), (4, 3), (5, 3), (6, 3), (1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (6, 6)\}$
- (vii) Let E_7 be the event of getting a multiple of 2 on the first die and a multiple of 3 on the second die i.e. 2 or 4 or 6 on first die and 3 or 6 on the second. $\therefore E_7 = \{(2, 3), (4, 3), (6, 3), (2, 6), (4, 6), (6, 6)\}$

Venn Diagram and Graphical Representation of Events

Events can be represented graphically using Venn Diagrams. This helps in a better understanding of the sample space and helps in computing the probability. The following diagram illustrates the use of a Venn Diagram. Consider the two events E_1 and E_2 of example 1, which have a sample space $S = \{1, 2, 3, 4, 5, 6\}$. The events are defined as: $E_1 = \{2, 4, 6\}$ and $E_2 = \{2, 3, 5\}$. The following Venn diagrams can be used to represent these events:



Venn Diagram

Please note the following points about the diagram:

- $E_1 \cap E_2 = \{2\}$
- $E_1 \cup E_2 = \{2, 3, 4, 5, 6\}$
- $S = \{1, 2, 3, 4, 5, 6\}$

You can learn more about Venn diagrams from references.

4.3 COMPUTING MATHEMATICAL PROBABILITY

Let there be 'n' exhaustive cases in a random experiment which are mutually exclusive as well as equally likely. Let 'm' out of them be favourable for the happening of an event A (say), then the probability of happening event A (denoted by $P(A)$) is defined as:

$$P(A) = \frac{\text{Number of favourable cases for event } A}{\text{Number of exhaustive cases}} = \frac{m}{n} \quad \dots(1)$$

Probability of non-happening of the event A is denoted by $P(\bar{A})$ and is defined as:

$$P(\bar{A}) = \frac{\text{Number of favourable cases for event } \bar{A}}{\text{Number of exhaustive cases}} = \frac{n - m}{n} = 1 - \frac{m}{n}$$

$$\text{So, } P(\bar{A}) = 1 - P(A)$$

$$\text{Or } P(A) + P(\bar{A}) = 1 \quad \dots(2)$$

Therefore, we conclude that the sum of the probabilities of happening an event and that of its complementary event is 1.

$$\text{Also, since } 0 \leq m \leq n, \text{ therefore } 0 \leq P(A) \leq 1 \quad \dots(3)$$

Remark: Probability of an impossible event is always zero and that of certain event is 1, e.g. probability of getting 7 when we throw a die is zero, as getting 7 here is an impossible event and probability of getting either of the six faces is 1, as it is a certain event.

This definition of probability fails if

- (i) The cases are not equally likely, e.g. probability of a candidate passing a test is not defined, as passing or failing in a test are not equally likely cases
- (ii) The number of exhaustive cases is indefinitely large, e.g. probability of drawing an integer say 2, from the set of integers i.e., $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ by definition, is $1/\infty = 0$. But, in actuality, it is not so, drawing 2 is not impossible, i.e. there are some chances of drawing 2. Hence, this definition fails here also.

Simple Problems On Probability

Now, let us give some examples so that you become familiar as to how and when this definition of probability is used.

Example 3: A bag contains 4 red, 5 black and 2 green balls. One ball is drawn from the bag. Find the probability that (i) It is a red ball, (ii) It is not black (iii) It is green or black

Solution: Let R_1, R_2, R_3, R_4 denote 4 red balls in the bag. Similarly, B_1, B_2, B_3, B_4, B_5 denote 5 black balls and G_1, G_2 denote two green balls in the bag. Then the sample space for drawing a ball is given by $\{R_1, R_2, R_3, R_4, B_1, B_2, B_3, B_4, B_5, G_1, G_2\}$

In these examples, we will use equation 1, which is given below:

$$P(A) = \frac{\text{Number of favourable cases for event } A}{\text{Number of exhaustive cases}}$$

- (i) Let A be the event of getting a red ball, then $A = \{R_1, R_2, R_3, R_4\}$
 $\therefore$ Number of favourable cases = 4, and
 Number of exhaustive cases = 11.
 So, $P(A) = 4/11$.
- (ii) Let B be the event that drawn ball is not black, then $B = \{R_1, R_2, R_3, R_4, G_1, G_2\}$
 $\therefore$ Number of favourable cases = 6
 Number of exhaustive cases = 11.
 So, $P(B) = 6/11$.
- (iii) Let C be the event that drawn ball is green or black, then $C = \{B_1, B_2, B_3, B_4, B_5, G_1, G_2\}$
 $\therefore$ Number of favourable cases = 7
 Number of exhaustive cases = 11.
 So, $P(C) = 7/11$.

Example 4: Three unbiased coins are tossed simultaneously. Find the probability of getting (i) at least two heads (ii) at most two heads (iii) all heads (iv) exactly one head (v) exactly one tail

Solution: The sample space in this case is $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$. (We have used equation 1 to answer this question)

- (i) Let E1 be the event of getting at least 2 heads, then
 $E1 = \{HHT, HTH, THH, HHH\}$
 $\therefore$ Number of favourable cases = 4
 Number of exhaustive cases = 8
 $P(E1) = 4/8 = 1/2$
- (ii) Let E2 be the event of getting at most 2 heads, then
 $E2 = \{TTT, TTH, THT, HTT, HHT, HTH, THH\}$
 $\therefore$ Number of favourable cases = 7
 Number of exhaustive cases = 8
 $P(E2) = 7/8$
- (iii) Let E3 be the event of getting all heads, then $E3 = \{HHH\}$
 $\therefore$ Number of favourable cases = 1
 Number of exhaustive cases = 8
 $P(E3) = 1/8$
- (iv) Let E4 be the event of getting exactly one head, then
 $E4 = \{HTT, THT, TTH\}$
 $P(E4) = 3/8$
- (v) Let E5 be the event of getting exactly one tail, then
 $E5 = \{HHT, HTH, THH\}$
 $P(E5) = 3/8$

Example 5: In an experiment of throwing two fair dice, find the probability of getting (i) a doublet (ii) sum 7 (iii) sum greater than 8 (iv) 3 on first die and a multiple of 2 on second die (v) prime number on the first die and odd prime on the second die.

Solution: The sample space has already been given in (vii) of Sec. 4.2. Here, the sample space contains 36 elements i.e. **number of exhaustive cases** is 36.

- (i) Let E_1 be the event of getting a doublet (i.e. same number on both dice), then $E_1 = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$.
 $\therefore$ Number of favourable cases = 6
 $P(E_1) = 6/36 = 1/6$
- (ii) Let E_2 be the event of getting sum 7, then
 $E_2 = \{(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)\}$
 $\therefore$ Number of favourable cases = 6
 $P(E_2) = 6/36 = 1/6$
- (iii) Let E_3 be the event of getting sum greater than 8, then
 $E_3 = \{(3, 6), (6, 3), (4, 5), (5, 4), (4, 6), (6, 4), (5, 5), (5, 6), (6, 5), (6, 6)\}$
 $\therefore$ Number of favourable cases = 10
 $P(E_3) = 10/36 = 5/18$
- (iv) Let E_4 be the event of getting 3 on first die and multiple of 2 on second die, then $E_4 = \{(3, 2), (3, 4), (3, 6)\}$
 $\therefore$ Number of favourable cases = 3
 $P(E_4) = 3/36 = 1/12$
- (v) Let E_5 be the event of getting prime number on first die and odd prime on second die, then
 $E_5 = \{(2, 3), (2, 5), (3, 3), (3, 5), (5, 3), (5, 5)\}$
 $\therefore$ Number of favourable cases = 6
 $P(E_5) = 6/36 = 1/6$

Example 6: Out of 52 well shuffled playing cards, one card is drawn at random. Find the probability of getting (i) a red card (ii) a face card (iii) a card of spade (iv) a card other than club (v) a king

Solution: Here, the number of exhaustive cases is 52 and a pack of playing cards contains 13 cards of each suit (spade, club, diamond, heart). (

- i) Let A be the event of getting a red card. We know that there are 26 red cards,
 $\therefore$ Number of favourable cases = 26
 Number of exhaustive cases = 52
 So, $P(A) = 26/52 = 1/2$
- ii) Let B be the event of getting a face card. We know that there are 12 face cards (jack, queen and king in each suit),
 $\therefore$ Number of favourable cases = 12
 Number of exhaustive cases = 52
 So, $P(B) = 12/52 = 3/13$
- iii) Let C be the event of getting a card of spades. We know that there are 13 cards of spades
 $\therefore$ Number of favourable cases = 13
 Number of exhaustive cases = 52
 So, $P(C) = 13/52 = 1/4$
- iv) Let D be the event of getting a card other than club. As there are 39 cards other than that of club,
 $\therefore$ Number of favourable cases = 39
 Number of exhaustive cases = 52
 So, $P(D) = 39/52 = 3/4$

- v) Let E be the event of getting a king. We know that there are 4 kings,
 $\therefore$ Number of favourable cases = 4
 Number of exhaustive cases = 52
 So, $P(E) = 4/52 = 1/13$

Example 7: Find the probability of getting 53 Sundays in a randomly selected non-leap year.

Solution: We know that there are 365 days in a non-leap year.

$365 \text{ days} / 7 = 52 \text{ weeks } 1/7 \text{ days, i.e. one non-leap year} = (52 \text{ complete weeks} + \text{one over day}).$

This over day may be one of the days Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday

So, the number of exhaustive cases = 7

Let A be the event of getting 53 Sundays. There will be 53 Sundays in a non-leap year if and only if the over day is Sunday.

$\therefore$ Number of favourable cases for event A = 1

$\therefore P(A) = 1/7$

Example 8: A single letter selected at random from the word 'STATISTICS'. What is the probability that it is a vowel?

Solution: Here, as the total number of letters in the word 'STATISTICS' is $n = 10$, and the number of vowels in the word is $m = 3$ (vowels are a, i, i),
 $\therefore$ The required probability = $3/10$

Example 9: Three horses A, B and C are in a race. A is twice as likely to win as B and B is twice as likely to win as C. What are the respective probabilities of their winning?

Solution: Let p be the probability that A wins the race. Probability that B wins the race = twice the probability of A's winning, i.e. $= 2(p) = 2p$; and probability of C's winning = twice the probability of B's winning $= 2(2p) = 4p$

Now, as the sum of the probability of happening an event and that of its complementary event(s) is 1. Here, the complementary of A is the happening of B or C] $\therefore p + 2p + 4p = 1$, and hence $p = 1/7$.

Thus, Probability of A winning the race = $1/7$; Probability of B winning the race = $2/7$ and probability of C winning the race = $4/7$.

Now, let us take up some problems on probability that are based on permutations/combinations. You may refer to these topics from the Programme Post-Graduate Diploma in Applied Statistics (PGDAST) in Course MST001: Foundation in Mathematics and Statistics, Unit 4.

Example 10: Out of 52 well shuffled playing cards, two cards are drawn at random. Find the probability of getting. (i) One red and one black (ii) Both cards of the same suit (iii) One jack and other king (iv) One red and the other of clubs.

Solution: Out of 52 playing cards, two cards can be drawn in ${}^{52}C_2$ ways i.e. $52 \times 51 / 2! = 26 \times 51$ ways. So, **Number of exhaustive cases = 26×51**

- (i) Let A be the event of getting one red and one black card, then the number of favourable cases for the event A are ${}^{26}C_1 \times {}^{26}C_1$ [As one red card out of 26 red cards can be drawn in ${}^{26}C_1$ ways and one black card out of 26 black cards can be drawn in ${}^{26}C_1$ ways.]

$\therefore P(A) = {}^{26}C_1 \times {}^{26}C_1 / (26 \times 51) = (26 \times 26) / (26 \times 51) = 26/51$

- (ii) Let B be the event of getting both the cards of the same suit and i.e. two cards of spades or two cards of clubs or 2 cards of diamonds or 2 cards of hearts.
 $\therefore$ Number of favourable cases for event B = ${}^{13}C_2 + {}^{13}C_2 + {}^{13}C_2 + {}^{13}C_2$
 $= 4 \times {}^{13}C_2 = 4 \times 13 \times 12 / 2! = 2 \times 13 \times 12$
 $P(B) = (2 \times 13 \times 12) / (26 \times 51) = 4/17$
- (iii) Let C be the event of getting a jack and a king.
 $\therefore$ Number of favourable cases ${}^4C_1 \times {}^4C_1 = (4 \times 4)$
 $P(C) = (4 \times 4) / (26 \times 51) = 8/663$
- (iv) Let D be the event of getting one red and one card of club. $\therefore$ Number of favourable cases $P(D) = {}^{26}C_1 \times {}^{13}C_1 = 26 \times 13$
 $P(D) = (26 \times 13) / (26 \times 51) = 13/51$

Example 11: If the letters of the word STATISTICS are arranged randomly, then find the probability that all three T's are together.

Solution: Let E be the event that selected word contains 3 T's together. There are 10 letters in the word STATISTICS. If we consider three T's as a single letter TTT, then we have 8 letters i.e. 1 'TTT'; 3 'S'; 1 'A'; 2 'I' and 1 'C'.

Number of possible arrangements with three T's coming together = $8! / (2! \cdot 3!)$
 (Please note you have 3 Ss and 2 Is in the word STATISTICS, therefore, $8!$ Is being divided by $3!$ (for 3 S) and $2!$ (for 2 I).

Number of favourable cases for event E = $8! / (2! \cdot 3!)$ and

Number of exhaustive cases = Total number of permutations of 10 letters in the word STATISTICS = $10! / (2! \cdot 3! \cdot 3!)$ [as out of 10 letters, 3 are T's, 2 are I's and 3 are S's]

The probability that all three T's are together in a random arrangement of letters of the word STATISTICS $P(A) = 8! / (2! \cdot 3!) \div 10! / (2! \cdot 3! \cdot 3!)$

$$P(A) = 8! \cdot (2! \cdot 3! \cdot 3!) / (2! \cdot 3!) \cdot 10! = 3 \times 2 / 10 \times 9 = 1/15$$

Example 12: In a lottery, one has to choose six numbers at random out of the numbers from 1 to 30. He/ she will get the prize only if all the six chosen numbers matched with the six numbers already decided by the lottery committee. Find the probability of winning the prize.

Solution: Out of 30 numbers, 6 can be drawn in ${}^{30}C_6$ ways, = $30! / (24! \cdot 6!)$

$$= (30 \times 29 \times 28 \times 27 \times 26 \times 25) / (6 \times 5 \times 4 \times 3 \times 2) = 593775 \text{ ways}$$

$$\therefore \text{Number of exhaustive cases} = 593775$$

Out of these 593775 ways, there is only one way to win the prize (i.e. choose those six numbers that are already fixed by the committee). Here, the number of favourable cases is 1.

Hence, Favourable cases 1

$$P(\text{winning the prize}) = 1 / \text{Exhaustive cases} = 1/593775$$

Check Your Progress 1

1) Write the sample space if we draw a card from a pack of 52 playing cards.

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- 2) If a die and a coin are tossed simultaneously, write the event of getting
- (i) head and prime number
 - (ii) tail and an even number
 - (iii) head and multiple of 3

- 3) If two coins are tossed, then find the probability of getting.

- (i) At least one head
- (ii) head and tail
- (iii) At most one head

- 4) If three dice are thrown, then find the probability of getting

- (i) triplet
- (ii) sum 5
- (iii) sum at least 17
- (iv) prime number on first die and odd prime number on second and third die.

- 5) Find the probability of getting 53 Mondays in a randomly selected leap year.

4.4 AXIOMS OF PROBABILITY

The axioms are fundamental to probability and provide us with an approach to probability, i.e., an axiomatic approach to probability. It defines the probability function as follows:

Definition: Let S be a sample space for a random experiment and A be an event which is subset of S , then $P(A)$ is called probability function if it satisfies the following axioms:

- (i) $P(A)$ is real and $P(A) \geq 0$
- (ii) $P(S) = 1$
- (iii) If $A_1, A_2, \dots$ is any finite or infinite sequence of disjoint events in S , then $P(A_1 \text{ or } A_2 \text{ or } \dots \text{ or } A_n) = P(A_1) + P(A_2) + \dots + P(A_n)$

Now, let us give some results using probability function. But before taking up these results, we discuss some statements with their meanings in terms of set theory. If A and B are two events, then in terms of set theory, we write:

- i) 'At least one of the events A or B occurs' as $A \cup B$
- ii) 'Both the events A and B occurs' as $A \cap B$
- iii) 'Neither A nor B occurs' as $\bar{A} \cap \bar{B}$

- iv) 'Event A occurs and B does not occur' as $A \cap \bar{B}$
- v) 'Exactly one of the events A or B occurs' as $(\bar{A} \cap B) \cup (A \cap \bar{B})$
- vi) 'Not more than one of the events A or B occurs' as $(\bar{A} \cap B) \cup (A \cap \bar{B}) \cup (\bar{A} \cap \bar{B})$

Similarly, you can write the meanings in terms of set theory for such statement in case of three or more events e.g. in case of three events A, B and C, happening of at least one of the events is written as $A \cup B \cup C$.

Some Results Using Probability Function:

1 Prove that probability of the impossible event is zero

Proof: Let S be the sample space and ϕ be the set of impossible event.

$$\therefore S \cup \phi = S$$

$$\Rightarrow P(S \cup \phi) = P(S)$$

$$\Rightarrow P(S) + P(\phi) = P(S) \quad [\text{By axiom (iii)}]$$

$$\Rightarrow 1 + P(\phi) = 1 \quad [\text{By axiom (ii)}]$$

$$\Rightarrow P(\phi) = 0$$

2 Probability of non-happening of an event A i.e. complementary event $\bar{A}$ of A is given by $P(\bar{A}) = 1 - P(A)$

Proof: If S is the sample space then

$$A \cup \bar{A} = S \quad [\because A \text{ and } \bar{A} \text{ are mutually disjoint events}]$$

$$\Rightarrow P(A \cup \bar{A}) = P(S)$$

$$\Rightarrow P(A) + P(\bar{A}) = P(S) \quad [\text{Using axiom (iii)}]$$

$$= 1 \quad [\text{Using axiom (ii)}]$$

$$\Rightarrow P(\bar{A}) = 1 - P(A)$$

3. Prove that

$$(i) P(A \cap \bar{B}) = P(A) - P(A \cap B)$$

$$(ii) P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

Proof

If S is the sample space and $A, B \subset S$ then

$$(i) A = (A \cap \bar{B}) \cup (A \cap B)$$

$$\Rightarrow P(A) = P((A \cap \bar{B}) \cup (A \cap B))$$

$$= P(A \cap \bar{B}) + P(A \cap B)$$

[Using axiom (iii) as $A \cap \bar{B}$ and $A \cap B$ are mutually disjoint]

$$\Rightarrow P(A \cap \bar{B}) = P(A) - P(A \cap B)$$

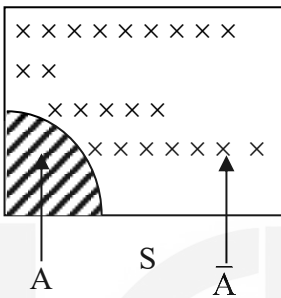
$$(ii) B = (A \cap B) \cup (\bar{A} \cap B)$$

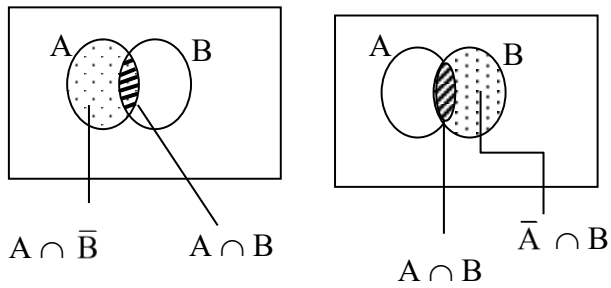
$$P(B) = P((A \cap B) \cup (\bar{A} \cap B))$$

$$= P(A \cap B) + P(\bar{A} \cap B)$$

[Using axiom (iii) as $A \cap B$ and $\bar{A} \cap B$ are mutually disjoint]

$$\Rightarrow P(\bar{A} \cap B) = P(B) - P(A \cap B)$$





Example 13: A, B and C are three mutually exclusive and exhaustive events associated with a random experiment. Find $P(A)$ given that :

$$P(B) = \frac{3}{4}P(A) \text{ and } P(C) = \frac{1}{3}P(B)$$

Solution:

As A, B and C are mutually exclusive and exhaustive events,

$$\therefore A \cup B \cup C = S$$

$$\Rightarrow P(A \cup B \cup C) = P(S)$$

$$\Rightarrow P(A) + P(B) + P(C) = 1 \quad \left[\because \text{using axiom (iii) as A, B, C are mutually disjoint events} \right]$$

$$\Rightarrow P(A) + \frac{3}{4}P(A) + \frac{1}{3}P(B) = 1$$

$$\Rightarrow P(A) + \frac{3}{4}P(A) + \frac{1}{3} \left(\frac{3}{4}P(A) \right) = 1$$

$$\Rightarrow \left(1 + \frac{3}{4} + \frac{1}{4} \right) P(A) = 1$$

$$\Rightarrow 2P(A) = 1$$

$$\Rightarrow P(A) = \frac{1}{2}$$

Example 14: If two dice are thrown, what is the probability that sum is

- greater than 9, and
- neither 10 or 12.

Solution:

$$\text{a) } P[\text{sum} > 9] = P[\text{sum} = 10 \text{ or sum} = 11 \text{ or sum} = 12]$$

$$= P[\text{sum} = 10] + P[\text{sum} = 11] + P[\text{sum} = 12]$$

[using axiom (iii)]

$$= \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{6}{36} = \frac{1}{6}$$

[$\therefore$ for sum = 10, there are three favourable cases (4, 6), (5, 5) and (6, 4). Similarly for sum = 11 and 12, there are two and one favourable cases respectively.]

Let A denotes the event for sum = 10 and B denotes the event for sum = 12,

$\therefore$ Required probability = $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B})$ [Using De- Morgan's law

(see Unit 1 of Course MST-001)]

$$= 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B)] \quad \text{[Using axiom (iii)]}$$

$$= 1 - \left[\frac{3}{36} + \frac{1}{36} \right] = 1 - \frac{4}{36} = 1 - \frac{1}{9} = \frac{8}{9}$$

4.5 ADDITION LAW

Addition Theorem on Probability for Two Events

Statement

Let S be the sample space of a random experiment, and events A and $B \subseteq S$ then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

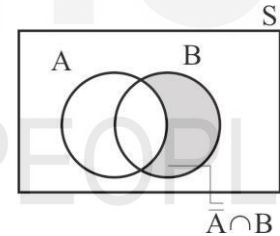
Proof: From the Venn diagram, we have

$$A \cup B = A \cup (\bar{A} \cap B)$$

$$\therefore P(A \cup B) = P(A) + P(\bar{A} \cap B)$$

[By axiom (iii) $\therefore$ A and $\bar{A} \cap B$ are mutually disjoint]

$$= P(A) + P(B) - P(A \cap B) \quad \text{[Refer to section 4.4]}$$



Hence proved

Corollary: If events A and B are mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B).$$

[This is known as the addition theorem for mutually exclusive events]

Proof: For any two events A and B, we know that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Now, if the events A and B are mutually exclusive then

$$A \cap B = \phi$$

Also, we know that probability of impossible event is zero i.e.

$$P(A \cap B) = P(\phi) = 0.$$

Hence,

$$P(A \cup B) = P(A) + P(B)$$

Similarly, for three non-mutually exclusive events A, B and C, we have

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

and for three mutually exclusive events A, B and C, we have

$$P(A \cup B \cup C) = P(A) + P(B) + P(C).$$

The result can similarly be extended for more than 3 events.

Applications of the Addition Theorem of Probability

Example 15: 25 lottery tickets are marked with first 25 numerals. A ticket is drawn at random. Find the probability that it is a multiple of 5 or 7.

Solution: Let A be the event that the drawn ticket bears a number multiple of 5 and B be the event that it bears a number multiple of 7.

Therefore,

$$A = \{5, 10, 15, 20, 25\}$$

$$B = \{7, 14, 21\}$$

$$\text{Here, as } A \cap B = \phi,$$

$\therefore$ A and B are mutually exclusive, and hence,

$$P(A \cup B) = P(A) + P(B) = \frac{5}{25} + \frac{3}{25} = \frac{8}{25}$$

Example 16: There are 40 pages in a book. A page is opened at random. Find the probability that the number of this opened page is a multiple of 3 or 5.

Solution Let A be the event that the number of the opened page is a multiple of 3, and B be the event that it is a multiple of 5.

$$\therefore A = \{3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39\},$$

$$B = \{5, 10, 15, 20, 25, 30, 35, 40\}, \text{ and}$$

$$A \cap B = \{15, 30\}$$

As A and B are non-mutually exclusive,

$$\therefore \text{the required probability} = P(A \cup B)$$

$$= P(A) + P(B) - P(A \cap B)$$

$$= \frac{13}{40} + \frac{8}{40} - \frac{2}{40} = \frac{19}{40}.$$

Check Your Progress 2

- 1) If A, B and C are any three events, write down the expressions in terms of set theory:
 - a) Only A occurs
 - b) A and B occur, but C does not
 - c) A, B and C all the three occur

- d) at least two occur
- e) exactly two do not occur
- f) none occurs

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- 2) Fourteen balls are serially numbered and placed in a bag. Find the probability that a ball is drawn bears a number multiple of 3 or 5.

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- 3) A card is drawn from a pack of 52 playing cards. Find the probability that it is either a king or a red card.

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- 4) Two dice are thrown together. Find the probability that the sum of the numbers turned up is either 6 or 8.

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4.6 CONDITIONAL PROBABILITY

We have discussed earlier that $P(A)$ represents the probability of happening event A for which the number of exhaustive cases is the number of elements in the sample space S . $P(A)$ dealt earlier was the unconditional probability. Here, we are going to deal with conditional probability.

Let us start with taking the following example:

Suppose a card is drawn at random from a pack of 52 playing cards. Let A be the event of drawing a black colour face card. Then $A = \{J_s, Q_s, K_s, J_c, Q_c, K_c\}$ and hence

$$P(A) = 6/52 = 3/26.$$

Let B be the event of drawing a card of spade i.e.

$$B = \{1_s, 2_s, 3_s, 4_s, 5_s, 6_s, 7_s, 8_s, 9_s, 10_s, J_s, Q_s, K_s\}.$$

If after a card is drawn from the pack of cards, we are given the information that card of spade has been drawn i.e., B has happened, then the probability of event

A will no more be $3/26$ because here in this case, we have the information that the card drawn is of spade (i.e. from amongst 13 cards) and hence there are 13 exhaustive cases and not 52. From amongst these 13 cards of spades, there are 3 black colour face cards and hence probability of having black colour face card given that it is a card of spades i.e. $P(A|B) = 3/13$, which is the conditional probability of A given that B has already happened.

Note: Here, the symbol ‘|’ used in $P(A|B)$ should be read as ‘given’ and not ‘upon’. $P(A|B)$ is the conditional probability of happening A given that B has already happened.

So, the conditional probability $P(A|B)$ is also the probability of happening A but here the information is given that the event B has already happened. $P(A|B)$ refers to the sample space B and not S.

Remark 1: $P(A|B)$ is meaningful only when $P(B) \neq 0$ i.e. when the event B is not an impossible event.

Multiplication Law of Probability

Statement: For two events A and B,

$$P(A \cap B) = P(A) P(B|A), P(A) > 0 \quad \dots (1)$$

$$= P(B) P(A|B), P(B) > 0, \quad \dots (2)$$

where $P(B|A)$ is the conditional probability of B given that A has already happened and $P(A|B)$ is the conditional probability of A given that B has already happened.

Proof: Let n be the number of exhaustive cases corresponding to the sample space S and m_1, m_2, m_3 be the number of favourable cases for events A, B and $A \cap B$ respectively.

$$\therefore P(A) = \frac{m_1}{n}, P(B) = \frac{m_2}{n}, P(A \cap B) = \frac{m_3}{n}$$

Now, as $B|A$ represents the event of happening B given that A has already happened and hence it refers to the sample space A ($\because$ we have with us the information that A has already happened) and thus the number of exhaustive cases for $B|A$ is m_1 (i.e. the number of cases favourable to “A relative to sample space S”). The number of cases favourable to $B|A$ is the number of those elements of B which are in A i.e. the number of favourable cases to $B|A$ is the number of favourable cases to “ $A \cap B$ relative to S”. So, the number of favourable cases to $B|A$ is m_3

$$\therefore P(B|A) = \frac{m_3}{m_1}$$

$$= \frac{m_3 / n}{m_1 / n} \quad \left[\begin{array}{l} \text{Dividing the numerator} \\ \text{and denominator by } n \end{array} \right]$$

$$= \frac{P(A \cap B)}{P(A)}, P(A) \neq 0$$

$$\Rightarrow P(A \cap B) = P(A) P(B|A), P(A) \neq 0$$

Similarly, you can prove that

$$P(A \cap B) = P(B) P(A|B), P(B) \neq 0.$$

This result can be extended for 3 or more events e.g. for three events A, B and C, we have

$$P(A \cap B \cap C) = P(A) P(B|A) P(C|A \cap B),$$

where $P(C|A \cap B)$ represents the probability of happening C given that A and B both have already happened.

4.7 INDEPENDENT EVENTS

Before defining the independent events, let us again consider the concept of conditional probability, taking the following example:

Suppose we draw a card from a pack of 52 playing cards, then probability of drawing a card of spades is $13/52$. Now, if we do not replace the card back and draw the next card. Then, the probability of drawing the second card 'a card of spades' given that the first card was spades would be $12/51$ and it is the conditional probability. Now, if the first card had been replaced back then this conditional probability would have been $13/52$. So, if sampling is done without replacement, the probability of second draw and that of subsequent draws made following the same way is affected but if it is done with replacement, then the probability of second draw and subsequent draws made following the same way remains unaltered.

In the example above, if the next draw is made with replacement, then the draw is not affected by the preceding draws. Let us now define independent events.

Independent Events

Events are said to be independent if happening or non-happening of any one event is not affected by the happening or non-happening of other events. For example, if a coin is tossed certain number of times, then happening of head in any trial is not affected by any other trial i.e. all the trials are independent.

Two events A and B are independent if and only if $P(B|A) = P(B)$ i.e. there is no relevance of giving any information. Here, if A has already happened, even then it does not alter the probability of B. e.g. Let A be the event of getting head in the 4th toss of a coin and B be the event of getting head in the 5th toss of the coin. Then the probability of getting head in the 5th toss is $1/2$, irrespective of the case whether we know or don't know the outcome of 4th toss, i.e. $P(B|A) = P(B)$.

Multiplicative Law for Independent Events:

If A and B are independent events, then

$$P(A \cap B) = P(A) P(B).$$

This is because if A and B are independent, then $P(B|A) = P(B)$ and hence the equation (1) discussed in the previous section becomes $P(A \cap B) = P(A) P(B)$.

Similarly, if A, B and C are three independent events, then

$$P(A \cap B \cap C) = P(A) P(B) P(C).$$

The result can be extended for more than three events also.

Remark 2: Mutually exclusive events can never be independent.

Proof: Let A and B be two mutually exclusive events with positive probabilities (i.e. $P(A) > 0$, and $P(B) > 0$)

$$\therefore P(A \cap B) = 0 \quad [\because A \cap B = \phi, \text{ as the events are mutually exclusive}]$$

Also, by multiplication law of probability, we have

$$P(A \cap B) = P(A) P(B|A), P(A) \neq 0.$$

$$\therefore 0 = P(A) P(B|A).$$

$$\text{Now as } P(A) \neq 0, \therefore P(B|A) = 0$$

$$\text{But } P(B) \neq 0 \quad [\because P(B) > 0]$$

$$\therefore P(B|A) \neq P(B),$$

Hence, A and B are not independent.

Result: If events A and B are independent, then prove that

- (i) A and $\bar{B}$ are independent
- (ii) $\bar{A}$ and B are independent
- (iii) $\bar{A}$ and $\bar{B}$ are independent

Proof:

(i) We know that

$$\begin{aligned} P(A \cap \bar{B}) &= P(A) - P(A \cap B) && [\text{Already proved in this Unit}] \\ &= P(A) - P(A)P(B) && [\because \text{events A and B are independent.}] \\ &= P(A)(1 - P(B)) \\ &= P(A)P(\bar{B}) \end{aligned}$$

$\Rightarrow$ Events A and $\bar{B}$ are independent

(ii) We know that

$$\begin{aligned} P(\bar{A} \cap B) &= P(B) - P(A \cap B) && [\text{Already proved in this unit}] \\ &= P(B) - P(A)P(B) && [\because \text{events A and B are independent.}] \\ &= P(B)(1 - P(A)) \\ &= P(B)P(\bar{A}) \\ &= P(\bar{A})P(B) \end{aligned}$$

$\Rightarrow$ Events $\bar{A}$ and B are independent

$$(iii) P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) \quad [\text{By De-Morgan's law}]$$

$$= 1 - P(A \cup B) \quad [\because P(E) + P(\bar{E}) = 1]$$

$$= 1 - [P(A) + P(B) - P(A \cap B)] \quad [\text{Using addition law on probability}]$$

$$= 1 - [P(A) + P(B) - P(A)P(B)] \quad [\because \text{events A and B are independent.}]$$

$$= 1 - P(A) - P(B) + P(A)P(B)$$

$$= (1 - P(A)) - P(B)(1 - P(A))$$

$$= (1 - P(A))(1 - P(B))$$

$$= P(\bar{A})P(\bar{B})$$

$\Rightarrow$ Events $\bar{A}$ and $\bar{B}$ are independent

Now let us take up some examples on conditional probability, multiplicative law and independent events:

Example 17: A die is rolled. If the outcome is a number greater than 3, what is the probability that it is a prime number?

Solution: The sample space of the experiment is

$$S = \{1, 2, 3, 4, 5, 6\}$$

Let A be event that the outcome is a number greater than 3 and B be the event that it is a prime number.

$$\therefore A = \{4, 5, 6\}, B = \{2, 3, 5\} \text{ and hence } A \cap B = \{5\}.$$

$$\Rightarrow P(A) = 3/6, P(B) = 3/6, P(A \cap B) = 1/6.$$

Now, the required probability = $P(B|A)$

$$= \frac{P(A \cap B)}{P(A)} \quad [\text{Refer to equation 1}]$$

$$= \frac{1/6}{3/6} = \frac{1}{3}$$

Example 18: A couple has 2 children. What is the probability that both children are boys, if it is known that?

- (i) younger child is a boy
- (ii) older child is a boy
- (iii) at least one of them is boy

Solution: Let B_i, G_i denote that i^{th} birth is of boy and girl respectively, $i = 1, 2$.

Then, for a couple having two children, the sample space is

$$S = \{B_1B_2, B_1G_2, G_1B_2, G_1G_2\}$$

Let A be the event that both children are boys then

$$A = \{B_1 B_2\}$$

(i) Let B be the event of getting younger child as boy i.e.

$$B = \{B_1 B_2, G_1 B_2\}. \text{ Hence } A \cap B = \{B_1 B_2\}$$

$$\therefore \text{required probability } P(A | B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{2/4} = \frac{1}{2}$$

(ii) Let C be the event of getting older child as boy, then $C = \{B_1 B_2, B_1 G_2\}$

$$\text{and hence } A \cap C = \{B_1 B_2\}.$$

$$\therefore \text{required probability } P(A | C) = \frac{P(A \cap C)}{P(C)} = \frac{1/4}{2/4} = \frac{1}{2}.$$

(iii) Let D be the event of getting at least one of the children as boy, then

$$D = \{B_1 B_2, B_1 G_2, G_1 B_2\} \text{ and hence}$$

$$A \cap D = \{B_1 B_2\}.$$

$$\therefore \text{required probability } P(A | D) = \frac{P(A \cap D)}{P(D)} = \frac{1/4}{3/4} = \frac{1}{3}$$

Example 19: An urn contains 4 red and 7 blue balls. Two balls are drawn one by one without replacement. Find the probability of getting 2 red balls.

Solution: Let A be the event that first ball drawn is red and B be the event that the second ball drawn is red.

$$\therefore P(A) = 4/11 \text{ and } P(B|A) = 3/10 \left[\begin{array}{l} \because \text{it is given that one red ball} \\ \text{has already been drawn} \end{array} \right]$$

$$\therefore \text{The required probability} = P(A \cap B)$$

$$= P(A) P(B|A)$$

$$= \left(\frac{4}{11}\right) \left(\frac{3}{10}\right) = \frac{6}{55}$$

Example 20: Three cards are drawn one by one without replacement from a well shuffled pack of 52 playing cards. What is the probability that first card is jack, second is queen and the third is again a jack.

Solution: Define the following events

E_1 be the event of getting a jack in the first draw,

E_2 be the event of getting a queen in second draw, and

E_3 be the event of getting a jack in third draw,

$$\therefore \text{Required probability} = P(E_1 \cap E_2 \cap E_3)$$

$$= P(E_1)P(E_2 | E_1)P(E_3 | E_1 \cap E_2)$$

[$\because$ cards are drawn without replacement and
hence the events are not independent]

$$= \frac{4}{52} \times \frac{4}{51} \times \frac{3}{50} = \frac{1}{13} \times \frac{2}{17} \times \frac{1}{25} = \frac{2}{5525}.$$

Example 21: Solve the following for the events A and B.

- (i) If A and B are independent events with $P(A \cup B) = 0.8$ and $P(B) = 0.4$, then find $P(A)$.
- (ii) If A and B are independent events with $P(A) = 0.2$, $P(B) = 0.5$, then find $P(A \cup B)$.
- (iii) If A and B are independent events and $P(A) = 0.4$ and $P(B) = 0.3$, then find $P(A|B)$ and $P(B|A)$.
- (iv) If A and B are independent events with $P(A) = 0.4$ and $P(B) = 0.2$, then find

$$P(\bar{A} \cap B), P(A \cap \bar{B}), P(\bar{A} \cap \bar{B})$$

Solution:

- (i) We are given

$$P(A \cup B) = 0.8, P(B) = 0.4.$$

We know that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad [\text{By Addition theorem of probability}]$$

$$= P(A) + P(B) - P(A)P(B) \quad [\because \text{events A and B are independent}]$$

$$\Rightarrow 0.8 = P(A) + 0.4 - 0.4P(A)$$

$$\Rightarrow 0.4 = (1 - 0.4) P(A)$$

$$= 0.6 P(A)$$

$$\Rightarrow P(A) = 0.4/0.6 = 2/3$$

- (ii) We are given that $P(A) = 0.2$, $P(B) = 0.5$.

We know that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad [\text{Addition theorem of probability}]$$

$$= P(A) + P(B) - P(A)P(B) \quad [\because \text{events A and B are independent}]$$

$$= 0.2 + 0.5 - 0.2 \times 0.5$$

$$= 0.7 - 0.10 = 0.6$$

- (iii) We are given that $P(A) = 0.4$, $P(B) = 0.3$.

Now, as A and B are independent events,

$$\therefore P(A \cap B) = P(A)P(B)$$

$$= 0.4 \times 0.3 = 0.12$$

And hence from conditional probability, we have

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.12}{0.3} = \frac{12}{30} = \frac{2}{5} = 0.4,$$

$$\text{and } P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.12}{0.4} = \frac{12}{40} = \frac{3}{10} = 0.3.$$

(iv) We are given that $P(A) = 0.4$, $P(B) = 0.2$.

We know that if two events A and B are independent then

$\bar{A}$ and B; A and $\bar{B}$; $\bar{A}$ and $\bar{B}$ are also independent events.

$\therefore P(\bar{A} \cap B) = P(\bar{A})P(B)$ [Using the concept of independent events]

$$= (1 - P(A))(P(B))$$

$$= (1 - 0.4)(0.2)$$

$$= (0.6)(0.2)$$

$$= 0.12$$

$$P(A \cap \bar{B}) = P(A)P(\bar{B}) \quad [\because A \text{ and } \bar{B} \text{ are independent}]$$

$$= (0.4)(1 - 0.2) = 0.32$$

$$P(\bar{A} \cap \bar{B}) = P(\bar{A})P(\bar{B}) = (1 - 0.4)(1 - 0.2) = (0.6)(0.8) = 0.48.$$

Example 22: Three unbiased coins are tossed simultaneously. In which of the following cases are the events A and B independent?

(i) A be the event of getting exactly one head

B be the event of getting exactly one tail

(ii) A be the event that first coin shows head

B be the event that third coin shows tail

(iii) A be the event that shows exactly two tails

B be the event that third coin shows head

Solution: When three unbiased coins are tossed simultaneously, then the sample space is given by

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

(i) $A = \{HTT, THT, TTH\}$

$$B = \{HHT, HTH, THH\}$$

$$A \cap B = \{\} = \phi$$

$$\therefore P(A) = \frac{3}{8}, P(B) = \frac{3}{8}, P(A \cap B) = \frac{0}{8} = 0$$

$$\text{Hence, } P(A)P(B) = \frac{3}{8} \times \frac{3}{8} = \frac{9}{64}$$

$$\Rightarrow P(A \cap B) \neq P(A)P(B)$$

$\Rightarrow$ Events A and B are not independent.

$$(ii) A = \{HHH, HHT, HTH, HTT\}$$

$$B = \{HHT, HTT, THT, TTT\}$$

$$A \cap B = \{HHT, HTT\}$$

$$\therefore P(A) = \frac{4}{8} = \frac{1}{2}, P(B) = \frac{4}{8} = \frac{1}{2}, P(A \cap B) = \frac{2}{8} = \frac{1}{4}$$

$$P(A)P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = P(A \cap B)$$

$\Rightarrow$ Events A and B are independent.

$$(iii) A = \{HTT, THT, TTH\}$$

$$B = \{HHH, HTH, THH, TTH\}$$

$$A \cap B = \{TTH\}$$

$$P(A) = \frac{3}{8}, P(B) = \frac{4}{8} = \frac{1}{2}, P(A \cap B) = \frac{1}{8}.$$

$$\text{Hence, } P(A)P(B) = \frac{3}{8} \times \frac{1}{2} = \frac{3}{16} \neq P(A \cap B)$$

$\Rightarrow$ Events A and B are not independent.

Example 23: Two cards are drawn from a pack of cards in succession with replacement of first card. Find the probability that both are the cards of 'hearts'.

Solution: Let A be the event that the first card drawn is a hearts card and B be the event that second card is a hearts card.

As the cards are drawn with replacement,

$\therefore$ A and B are independent and hence the required probability

$$= P(A \cap B) = P(A)P(B) = \left(\frac{13}{52}\right)\left(\frac{13}{52}\right) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}.$$

Example 24: A class consists of 10 boys and 40 girls. 5 of the students are rich and 15 students are brilliant. Find the probability of selecting a brilliant rich boy.

Solution: Let A be the event that the selected student is brilliant, B be the event that he/she is rich and C be the event that the student is boy.

$$\therefore P(A) = \frac{15}{50}, P(B) = \frac{5}{50}, P(C) = \frac{10}{50} \text{ and hence}$$

the required probability = $P(A \cap B \cap C)$

$$= P(A)P(B)P(C) \quad [\because A, B \text{ and } C \text{ are independent}]$$

$$= \left(\frac{15}{50}\right)\left(\frac{5}{50}\right)\left(\frac{10}{50}\right) = \left(\frac{3}{10}\right)\left(\frac{1}{10}\right)\left(\frac{1}{5}\right) = \frac{3}{500}$$

Check Your Progress 3

- 1) A card is drawn from a well-shuffled pack of cards. If the card drawn is a face card, what is the probability that it is a king?

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- 2) Two cards are drawn one by one without replacement from a well shuffled pack of 52 cards. What is the probability that both the cards are red?

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- 3) A bag contains 10 good and 4 defective items, two items are drawn one by one without replacement. What is the probability that first drawn item is defective and the second one is good?

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- 4) The odds in favour of passing driving test by a person X are 3 : 5 and odds in favour of passing the same test by another person Y are 3 : 2. What is the probability that both will pass the test?

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4.8 BAYES' THEOREM

One of the important theorems of probability that is useful in data science is known as Bayes' theorem, given by Thomas Bayes, a British Mathematician.. In this unit we state this theorem without proof.

Statement: Let S be the sample space and $E_1, E_2, \dots, E_n$ be n mutually exclusive and exhaustive events with $P(E_i) \neq 0$; $i = 1, 2, \dots, n$. Let A be any event which is a sub-set of $E_1 \cup E_2 \cup \dots \cup E_n$ (i.e. at least one of the events $E_1, E_2, \dots, E_n$) with $P(A) > 0$ [Notice that up to this line the statement is the same as that of law of total probability], then

$$P(E_i | A) = \frac{P(E_i)P(A|E_i)}{P(A)}, i = 1, 2, \dots, n$$

where $P(A) = P(E_1) P(A|E_1) + P(E_2) P(A|E_2) + \dots + P(E_n) P(A|E_n)$.

Applications Of Bayes' Theorem

Example 25(a): There are two bags. First bag contains 5 red, 6 white balls and the second bag contains 3 red, 4 white balls. One bag is selected at random and a ball is drawn from it. What is the probability that it is (i) red, (ii) white.

Solution: Let E_1 be the event that first bag is selected and E_2 be the event that second bag is selected.

$$\therefore P(E_1) = P(E_2) = \frac{1}{2}.$$

(i) Let R be the event of getting a red ball from the selected bag.

$$\therefore P(R | E_1) = \frac{5}{11}, \text{ and } P(R | E_2) = \frac{3}{7}.$$

Thus, the required probability is given by

$$P(R) = P(E_1)P(R | E_1) + P(E_2)P(R | E_2)$$

$$\begin{aligned} &= \frac{1}{2} \times \frac{5}{11} + \frac{1}{2} \times \frac{3}{7} \\ &= \frac{5}{22} + \frac{3}{14} = \frac{35 + 33}{154} = \frac{68}{154} = \frac{34}{77} \end{aligned}$$

(ii) Let W be the event of getting a white ball from the selected bag.

$$\therefore P(W | E_1) = \frac{6}{11}, \text{ and } P(W | E_2) = \frac{4}{7}.$$

Thus, the required probability is given by

$$P(W) = P(E_1)P(W | E_1) + P(E_2)P(W | E_2)$$

$$= \frac{1}{2} \times \frac{6}{11} + \frac{1}{2} \times \frac{4}{7} = \frac{3}{11} + \frac{2}{7} = \frac{21 + 22}{77} = \frac{43}{77}.$$

Example 25(b): Let us consider the problem given in Example 25(a) after replacing the question asked therein (i.e. the last sentence) by the following question:

If it is found to be red, what is the probability of?

i) selecting the first bag

ii) selecting the second bag

Solution: First, we have to give the solution exactly as given for Example 25(a). After that, we are to proceed as follows:

i) Probability of selecting the first bag given that the ball drawn is red

$$= P(E_1 | R)$$

$$= \frac{P(E_1)P(R | E_1)}{P(R)} \quad \left[\begin{array}{l} \text{Applying Bayes'} \\ \text{theorem} \end{array} \right]$$

$$= \frac{\frac{1}{2} \times \frac{5}{11}}{\frac{34}{77}} = \frac{5}{22} \times \frac{77}{34} = \frac{35}{68}$$

ii) Probability of selecting the second bag given that the ball drawn is red

$$P(E_2|R) = \frac{P(E_2)P(R|E_2)}{P(R)} = \frac{\frac{1}{2} \times \frac{3}{7}}{\frac{34}{77}} = \frac{3}{14} \times \frac{77}{34} = \frac{33}{68}$$

Example 26(a): A factory produces certain type of output by 3 machines. The respective daily production figures are - machine X: 3000 units, machine Y: 2500 units and machine Z: 4500 units. Past experience shows that 1% of the output produced by machine X is defective. The corresponding fractions of defectives for the other two machines are 1.2 and 2 percent respectively. An item is drawn from the day's production. What is the probability that it is defective?

Solution: Let E_1 , E_2 and E_3 be the events that the drawn item is produced by machine X, machine Y and machine Z, respectively. Let A be the event that the drawn item is defective.

As the total number of units produced by all the machines is

$$3000 + 2500 + 4500 = 10000,$$

$$\therefore P(E_1) = \frac{3000}{10000} = \frac{3}{10}, P(E_2) = \frac{2500}{10000} = \frac{1}{4}, P(E_3) = \frac{4500}{10000} = \frac{9}{20}.$$

$$P(A|E_1) = \frac{1}{100} = 0.01, P(A|E_2) = \frac{1.2}{100} = 0.012, P(A|E_3) = \frac{2}{100} = 0.02.$$

Thus, the required probability = Probability that the drawn item is defective

$$\begin{aligned} &= P(A) \\ &= P(E_1) P(A|E_1) + P(E_2) P(A|E_2) + P(E_3) P(A|E_3) \\ &= \frac{3}{10} \times 0.01 + \frac{1}{4} \times 0.012 + \frac{9}{20} \times 0.02 \\ &= \frac{3}{1000} + \frac{3}{1000} + \frac{9}{1000} \\ &= \frac{15}{1000} = 0.015. \end{aligned}$$

Example 26(b): Consider the problem given in Example 26(a) after replacing the question asked therein (i.e. the last sentence) by the following question:

If the drawn item is found to be defective, what is the probability that it has been produced by machine Y?

Solution: Proceed exactly in the manner the Example 26(a) has been solved and then as under:

Probability that the drawn item has been produced by machine Y, given that it is defective

$$\begin{aligned}
 &= P(E_2 | A) \\
 &= \frac{P(E_2)P(A | E_2)}{P(A)} \quad \left[\begin{array}{l} \text{Applying Bayes'} \\ \text{theorem} \end{array} \right] \\
 &= \frac{\frac{1}{4} \times 0.012}{0.015} = \frac{0.003}{0.015} = \frac{1}{5}.
 \end{aligned}$$

Example 27(a): There are two coins-one unbiased and the other two-headed, otherwise they are identical. One of the coins is taken at random without seeing it and tossed. What is the probability of getting a head?

Solution: Let E_1 and E_2 be the events of selecting the unbiased coin and the two-headed coin, respectively. Let A be the event of getting head on the tossed coin.

$$\therefore P(E_1) = \frac{1}{2}, P(E_2) = \frac{1}{2} \quad [\because \text{selection of each of the coin is equally likely}]$$

$$P(A | E_1) = \frac{1}{2} \quad [\because \text{if it is unbiased coin, then head and tail are equally likely}]$$

$$P(A | E_2) = 1 \quad [\because \text{if it is two-headed coin, then getting the head is certain}]$$

Thus, the required probability = $P(A)$

$$\begin{aligned}
 &= P(E_1) P(A | E_1) + P(E_2) P(A | E_2) \\
 &= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times 1 \\
 &= \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.
 \end{aligned}$$

Example 27(b): Consider the problem given in Example 27(a), after replacing the question asked therein (i.e. the last sentence) by the following question:

If head turns up, what is the probability that

- i) it is the two-headed coin
- ii) it is the unbiased coin.

Solution: First give the solution of Example 27(a) and then proceed as under:

Now, the probability that the tossed coin is two-headed given that head turned up

$$\begin{aligned}
 &= P(E_2 | A) \\
 &= \frac{P(E_2)P(A | E_2)}{P(A)} \quad \left[\begin{array}{l} \text{Applying Bayes'} \\ \text{theorem} \end{array} \right]
 \end{aligned}$$

$$= \frac{\frac{1}{2} \times 1}{\frac{3}{4}} = \frac{1}{2} \times \frac{4}{3} = \frac{2}{3}.$$

ii) The probability that the tossed coin is unbiased given that head turned up

$$= P(E_1 | A)$$

$$= \frac{P(E_1)P(A | E_1)}{P(A)} \quad \left[\begin{array}{l} \text{Applying Bayes' } \\ \text{theorem} \end{array} \right]$$

$$= \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{3}{4}} = \frac{1}{4} \times \frac{4}{3} = \frac{1}{3}$$

Check Your Progress 4

- 1) A bag contains 4 red and 5 white balls. Another bag contains 2 red and 3 white balls. A ball is drawn from the first bag and is transferred to the second bag. A ball is then drawn from the second bag and is found to be red, what is the probability that red ball was transferred from first to second bag?

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- 2) An insurance company insured 1000 scooter drivers, 3000 car drivers and 6000 truck drivers. The probabilities that scooter, car and truck drivers meet an accident are 0.02, 0.04, 0.25, respectively. One of the insured persons meets with an accident. What is the probability that he is a

(i) car driver

(ii) truck driver

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- 3) By examining the chest X-ray, the probability that T.B is detected when a person is actually suffering from T.B. is 0.99. The probability that the doctor diagnoses incorrectly that a person has T.B. on the basis of X-ray is 0.002. In a certain city, one in 1000 persons suffers from T.B. A person is selected at random and is diagnosed to have T.B., what is the chance that he actually has T.B.?

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4.9 SUMMARY

In this unit, we have discussed:

1. How to create a sample space for a problem.
2. Define an event and create its instances.
3. How to define probability.
4. What are the axioms of probability, and how can these axioms be used
5. How to use the addition law, conditional probability and independent events.
6. What is Bayes' theorem and how can it be used to compute probability.

4.10 SOLUTIONS / ANSWERS

Check Your Progress 1

1) Let suffices C, D, S, H denote that corresponding card is a club, diamond, spade, heart, respectively then sample space of drawing a card can be written as

$$\{1_C, 2_C, 3_C, \dots, 9_C, 10_C, J_C, Q_C, K_C, \\ 1_D, 2_D, 3_D, \dots, 9_D, 10_D, J_D, Q_D, K_D, \\ 1_S, 2_S, 3_S, \dots, 9_S, 10_S, J_S, Q_S, K_S, \\ 1_H, 2_H, 3_H, \dots, 9_H, 10_H, J_H, Q_H, K_H, \}$$

2) If a die and a coin is thrown then sample space is

$$\{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$$

- (i) Let A be the event of getting head and prime number, then $A = \{H2, H3, H5\}$
- (ii) Let B be the event of getting tail and even number, then $B = \{T2, T4, T6\}$
- (iii) Let C be the event of getting head and multiple of 3, then $C = \{H3, H6\}$

3) When two coins are tossed simultaneously then sample space is

$$S = \{HH, HT, TH, TT\}$$

- (i) Let A be the event of getting at least one head, then $A = \{HH, HT, TH\}$ and $P(A) = \frac{3}{4}$

- (ii) Let B = event of getting both head and tail, then $B = \{HT, TH\}$ and $P(B) = \frac{2}{4} = \frac{1}{2}$

- (iii) Let C be the event of getting at most one head, then $C = \{TT, TH, HT\}$ and $P(C) = \frac{3}{4}$

4) When 3 dice are thrown, then the sample space is

$$S = \{ (1, 1, 1), (1, 1, 2), (1, 1, 3), \dots, (1, 1, 6), \\ (1, 2, 1), (1, 2, 2), (1, 2, 3), \dots, (1, 2, 6), \\ (1, 3, 1), (1, 3, 2), (1, 3, 3), \dots, (1, 3, 6), \\ \dots \\ (6, 6, 1), (6, 6, 2), (6, 6, 3), \dots, (6, 6, 6) \}$$

Number of elements in the sample space = $6 \times 6 \times 6 = 216$

$\left[\because \text{We have to fill up 3 positions } (.,.,.) \text{ and each position can be filled with 6 options, } \therefore \text{this can be done in } 6 \times 6 \times 6 = 216 \text{ ways} \right]$

- (i) Let A be the event of getting triplet i.e. same number on each die.

$$\therefore A = \{(1, 1, 1), (2, 2, 2), (3, 3, 3), (4, 4, 4), (5, 5, 5), (6, 6, 6)\}$$

$$\text{and hence } P(A) = \frac{6}{216} = \frac{1}{36}$$

- (ii) Let B be the event of getting sum 5

$$\therefore B = \{(1, 1, 3), (1, 3, 1), (3, 1, 1), (1, 2, 2), (2, 1, 2), (2, 2, 1)\}$$

$$\text{and } P(B) = \frac{6}{216} = \frac{1}{36}$$

- (iii) Let C be the event of getting sum at least 17 i.e. sum 17 or 18

$$\therefore C = \{(5, 6, 6), (6, 5, 6), (6, 6, 5), (6, 6, 6)\}$$

$$\text{and hence } P(C) = \frac{4}{216} = \frac{1}{54}$$

- (iv) Let D be the event of getting prime number on first die and odd prime number on second and third die. i.e. first die can show 2 or 3 or 5 and second, third dice can show 3 or 5

$$\therefore D = \{(2, 3, 3), (2, 3, 5), (2, 5, 3), (2, 5, 5), (3, 3, 3), (3, 3, 5), \\ (3, 5, 3), (3, 5, 5), (5, 3, 3), (5, 3, 5), (5, 5, 3), (5, 5, 5)\}$$

$$\text{and hence } P(D) = \frac{12}{216} = \frac{1}{18}$$

5) We know that there are 366 days in a leap year. i.e. $\frac{365}{7} = 52\frac{2}{7}$ weeks

i.e. one leap year = (52 complete weeks + two over days)

i.e. 52 complete weeks and two over days. These two over days may be

- (i) Sunday and Monday
- (ii) Monday and Tuesday
- (iii) Tuesday and Wednesday
- (iv) Wednesday and Thursday
- (v) Thursday and Friday
- (vi) Friday and Saturday
- (vii) Saturday and Sunday

$\therefore$ Number of exhaustive Cases = 7

Let A be the event of getting 53 Mondays

There will be 53 Mondays in a leap year if and only if these two over days are

“Sunday and Monday” or “Monday and Tuesday”

$\therefore$ Number of favourable cases for event A are 2

$$\text{and } P(A) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{2}{7}$$

Check Your Progress 2

1) a) $A \cap \bar{B} \cap \bar{C}$

b) $A \cap B \cap \bar{C}$

c) $A \cap B \cap C$

d) $(A \cap B \cap \bar{C}) \cup (A \cap \bar{B} \cap C) \cup (\bar{A} \cap B \cap C) \cup (A \cap B \cap C)$

e) $(\bar{A} \cap \bar{B} \cap C) \cup (\bar{A} \cap B \cap \bar{C}) \cup (A \cap \bar{B} \cap \bar{C})$

f) $\bar{A} \cap \bar{B} \cap \bar{C}$

- 2) Let A be the event that the drawn ball bears a number multiple of 3 and B be the event that it bears a number multiple of 5, then

$$A = \{3, 6, 9, 12\} \text{ and } B = \{5, 10\}$$

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 4/14 + 2/14 - 0 = 3/7 \end{aligned}$$

- 3) Let A be the event of getting a card of king and B be the event of getting a red card.

$\therefore$ the required probability

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad [\text{By addition theorem}]$$

$$= \frac{4}{52} + \frac{26}{52} - \frac{2}{52} \left[\because \text{there are two cards which are both king as well as red} \right]$$

$$= \frac{4}{52} + \frac{26}{52} - \frac{2}{52} = \frac{4+26-2}{52} = \frac{28}{52} = \frac{7}{13}$$

- 4) Here, the number of exhaustive cases = $6 \times 6 = 36$.

Let A be the event that the sum is 6 and B be the event that the sum is 8.

$$\therefore A = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}, \text{ and}$$

$$B = \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}.$$

Here, as $A \cap B = \phi$,

$\therefore$ A and B are mutually exclusive and hence

the required probability = $P(A \cup B) = P(A) + P(B)$

$$= \frac{5}{36} + \frac{5}{36} = \frac{10}{36} = \frac{5}{18}$$

Check Your Progress 3

- 1) Let A be the event that the card drawn is face card and B be the event that it is a king.

$$\therefore A = \{J_s, Q_s, K_s, J_h, Q_h, K_h, J_d, Q_d, K_d, J_c, Q_c, K_c\}$$

$$B = \{K_s, K_h, K_c, K_d\}$$

$$\Rightarrow A \cap B = \{K_s, K_h, K_c, K_d\}$$

$$\Rightarrow P(A) = \frac{12}{52}, P(B) = \frac{4}{52}, \text{ and } P(A \cap B) = \frac{4}{52}.$$

$$\therefore \text{the required probability} = P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{4/52}{12/52} = \frac{4}{12} = \frac{1}{3}.$$

- 2) Let A be the event that first card drawn is red and B be the event that second card is red.

$$\therefore P(A) = \frac{26}{52} \text{ and } P(B|A) = \frac{25}{51}.$$

Thus, the desired probability = $P(A \cap B)$

$$\begin{aligned} &= P(A) P(B|A) \left[\begin{array}{l} \text{By multiplication} \\ \text{theorem} \end{array} \right] \\ &= \left(\frac{26}{52} \right) \left(\frac{25}{51} \right) = \frac{25}{102}. \end{aligned}$$

- 3) Let E_1 be the event of getting a defective item in first draw and E_2 be the event of getting a good item in the second draw.

$\therefore$ the required probability = $P(E_1 \cap E_2)$

$$\begin{aligned} &= P(E_1)P(E_2 | E_1) \left[\begin{array}{l} \text{By multiplication} \\ \text{theorem} \end{array} \right] \\ &= \frac{4}{14} \times \frac{10}{13} = \frac{20}{91}. \end{aligned}$$

- 4) Let A be the event that person X passes the test and B be the event that person Y passes the test.

$$\therefore P(A) = \frac{3}{3+5} = \frac{3}{8}$$

$$\text{and } P(B) = \frac{3}{3+2} = \frac{3}{5}$$

Now, as both the person take the test independently,

$\therefore$ the required probability = $P(A \cap B)$

$$= P(A)P(B) = \left(\frac{3}{8} \right) \left(\frac{3}{5} \right) = \frac{9}{40}$$

Check Your Progress 4

- 1) Let E_1 be the event that a red ball is drawn from the first bag and E_2 be the event that the drawn ball from the first bag is white. Let A be the event of drawing a red ball from the second bag after transferring the ball drawn from first bag into it.

$$\therefore P(E_1) = \frac{4}{9}, P(E_2) = \frac{5}{9}$$

$$P(A|E_1) = \frac{3}{6}, P(A|E_2) = \frac{2}{6}$$

$\therefore$ By law of total probability,

$$\begin{aligned} P(A) &= P(E_1)P(A|E_1) + P(E_2)P(A|E_2) \\ &= \frac{4}{9} \times \frac{3}{6} + \frac{5}{9} \times \frac{2}{6} = \frac{11}{27}. \end{aligned}$$

Thus, the required probability = $P(E_1|A)$

$$= \frac{P(E_1)P(A|E_1)}{P(A)} \quad \left[\begin{array}{l} \text{Applying Bayes'} \\ \text{theorem} \end{array} \right]$$

$$= \frac{\frac{4}{9} \times \frac{3}{6}}{\frac{11}{27}} = \frac{6}{11}$$

2) Total number of drivers = 1000 + 3000 + 6000 = 10000.

Let E_1, E_2, E_3 be the events that the selected insured person is a scooter driver, car driver, truck driver respectively. Let A be the event that a driver meets with an accident.

$$\therefore P(E_1) = \frac{1000}{10000} = \frac{1}{10}, P(E_2) = \frac{2000}{10000} = \frac{1}{5}, P(E_3) = \frac{6000}{10000} = \frac{6}{10} = \frac{3}{5}.$$

$$P(A|E_1) = 0.02 = \frac{2}{100}, P(A|E_2) = 0.04 = \frac{4}{100}, P(A|E_3) = 0.25 = \frac{25}{100}$$

$\therefore$ By total probability theorem, we have

$$\begin{aligned} P(A) &= P(E_1)P(A|E_1) + P(E_2)P(A|E_2) + P(E_3)P(A|E_3) \\ &= \frac{1}{10} \times \frac{2}{100} + \frac{1}{5} \times \frac{4}{100} + \frac{3}{5} \times \frac{25}{100} \\ &= \frac{1}{500} + \frac{4}{500} + \frac{75}{500} = \frac{80}{500} = \frac{4}{25} \end{aligned}$$

i) The required probability = $P(E_2|A)$

$$= \frac{P(E_2)P(A|E_2)}{P(A)} \quad \left[\begin{array}{l} \text{Applying Bayes'} \\ \text{theorem} \end{array} \right]$$

$$= \frac{\frac{1}{5} \times \frac{4}{100}}{\frac{4}{25}} = \frac{4}{500} \times \frac{25}{4} = \frac{1}{20}.$$

ii) The required probability = $P(E_3 | A)$

$$\begin{aligned}
 &= \frac{P(E_3)P(A | E_3)}{P(A)} \left[\begin{array}{l} \text{Applying Bayes' } \\ \text{theorem} \end{array} \right] \\
 &= \frac{\frac{3}{5} \times \frac{25}{100}}{\frac{4}{25}} \\
 &= \frac{75}{500} \times \frac{25}{4} = \frac{75}{80} = \frac{15}{16}.
 \end{aligned}$$

- 3) Let E_1 be the event that a person selected from the population of the city is suffering from the T.B. and E_2 be the event that he/she is not suffering from the T.B. Let A be the event that the selected person is diagnosed to have T.B. Therefore, according to given,

$$P(E_1) = \frac{1}{1000} = 0.001, \quad P(E_2) = 1 - 0.001 = 0.999 = \frac{999}{1000},$$

$$P(A | E_1) = 0.99 = \frac{99}{100}, \quad P(A | E_2) = 0.002 = \frac{2}{1000}.$$

$\therefore$ By total probabilities theorem, we have

$$\begin{aligned}
 P(A) &= P(E_1) P(A|E_1) + P(E_2) P(A|E_2) \\
 &= \frac{1}{1000} \times \frac{99}{100} + \frac{999}{1000} \times \frac{2}{1000} \\
 &= \frac{99}{100000} + \frac{1998}{1000000} = \frac{990 + 1998}{1000000} = \frac{2988}{1000000}
 \end{aligned}$$

Thus, the required probability = $P(E_1|A)$

$$\begin{aligned}
 &= \frac{P(E_1)P(A | E_1)}{P(A)} \\
 &= \frac{\frac{1}{1000} \times \frac{99}{100}}{\frac{2988}{1000000}} \\
 &= \frac{990}{2988} = \frac{55}{166}.
 \end{aligned}$$

4.11 REFERENCES

IGNOU Course Material

1. Post-Graduate Diploma In Applied Statistics (PGDAST), MST 003: Probability Theory, Block 1: Basic Concepts in Probability, UNIT 1: Measures of Dispersion (for Sections 4.2, 4.3 and related portions in Sections 4.0, 4.1, 4.9, 4.10 and Check Your Progress questions)
2. PGDAST, MST 003: Probability Theory, Block 1: Basic Concepts in Probability, UNIT 2: Different Approaches to Probability Theory (for Sections 4.4 and related portions in Sections 4.0, 4.1, 4.9, 4.10 and Check Your Progress questions)
3. PGDAST, MST 003: Probability Theory, Block 1: Basic Concepts in Probability, UNIT 3: Laws of Probability (for Sections 4.5 to Section 4.7 and related portions in Sections 4.0, 4.1, 4.9, 4.10 and Check Your Progress questions)
4. PGDAST, MST 003: Probability Theory, Block 1: Basic Concepts in Probability, UNIT 4: Bayes' Theorem (for Sections 4.8 and related portions in Sections 4.0, 4.1, 4.9, 4.10 and Check Your Progress questions)

Reference Book

5. "Introduction to Probability and Statistics for Engineers and Scientists", 3rd Edition, Sheldon M. Ross, Elsevier Academic Press, 2004